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Penalty approximation method for a class of elliptic variational inequality problems[☆]

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Abstract

In this paper, two existence results of solutions for a class of elliptic variational inequalities are obtained by considering the approximations of the solutions to a class of penalized differential equations.

Keywords: Elliptic variational inequality; Penalized differential equation; Approximation; Sobolev space; Mapping of class $(S)_+$

1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^N$ with a smooth boundary. Denote $\mathbf{X}$ and $\mathbf{K}$ by the Sobolev space $H_0^1(\Omega)$ and $\{u \in \mathbf{X} : u(x) \geq 0 \text{ a.e. in } \Omega\}$, respectively. Recall that $\mathbf{X}$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = \left\{\int_\Omega |\nabla u|^2\right\}^{1/2}$.

In this paper, we are concerned with the existence of solutions for the following variational inequality problem: find $u \in \mathbf{K}$ such that

$$\int_\Omega \nabla u \cdot \nabla(v-u)\,\mathrm{d}x \geq \lambda \int_\Omega u(v-u)\,\mathrm{d}x + \int_\Omega |u|^{\alpha-2}u(v-u)\,\mathrm{d}x + \int_\Omega f(x,u)(v-u)\,\mathrm{d}x, \quad \forall\, v \in \mathbf{K}, \tag{1}$$

where $1 < \alpha < 2$, $\lambda > 0$ is a parameter, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying some conditions to be given.

The variational inequality problems associated with elliptic types have attracted much attention in recent years due to interesting theoretical questions arising from those problems, and also to their direct applications in mechanics, engineering, differential equations, etc.; see for example [1–4] and the references therein, where variational and topological methods have been used to study the existence and the bifurcation of solutions.

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In this paper, instead of using topological methods, we try to use the penalty method from optimization theory to solve (1). Normally, penalty methods can be used to solve constrained optimization problems via solving one or a sequence of unconstrained optimization problems. In other words, the penalty methods attempt to solve a constrained optimization problem by the minimization of an unconstrained function or several unconstrained functions. The main motivation for the use of penalty methods is that of solving the constrained optimization problem by employing some unconstrained minimization algorithm. Penalty methods can also be used in solving variational inequality problems, see [5–7]. In [5], Auslender and Teboulle considered a new class of multiplier interior point methods for solving variational inequality problems. In [6] and [7], penalty methods have been used to provide a constructive approximation method if the variational inequality problem of a pseudomonotone operator is uniquely solvable.

In this paper, motivated by the methods of [6] and [7], we will get, in both cases that $\lambda = \lambda_1$ and $\lambda_k < \lambda < \lambda_{k+1}$ $(k \geq 1)$, a solution of Problem (1) by considering the strong limit of the solutions of penalized differential equations (see (7) below). We remark that the solutions for variational inequalities considered in [6,7] were the weak limits of the penalized differential equations.

We recall some concepts that we will use throughout this paper.

First we say that a nonlinear operator $T : \mathbf{X} \to \mathbf{X}^*$ is hemicontinuous if each real function $t \mapsto \langle T((1-t)u + tv), v - u\rangle$ with $u, v \in \mathbf{X}$ is continuous on $\mathbb{R}$; $T : \mathbf{X} \to \mathbf{X}^*$ is said to be monotone if $\langle T(u) - T(v), u - v\rangle \geq 0$ for $u, v \in \mathbf{X}$. $T : \mathbf{X} \to \mathbf{X}^*$ is said to be of class $(S)_+$ if, for $y_j \rightharpoonup y_0 \in \mathbf{X}$, $\limsup_{j\to\infty}\langle T(y_j), y_j - y_0\rangle \leq 0$ implies that $y_j \to y_0$ as $j \to \infty$.

We say that a bounded, hemicontinuous and monotone operator $\beta : \mathbf{X} \to \mathbf{X}^*$ is a penalty operator associated with $\mathbf{K}$ (see [7, p. 225]) if

$$\beta(u) = 0 \Leftrightarrow u \in \mathbf{K}.$$

Define operators $A, F : \mathbf{X} \to \mathbf{X}^*$ as for all $u, v \in \mathbf{X}$,

$$\begin{aligned} \langle A(u), v\rangle &:= \int_\Omega \nabla u \nabla v \mathrm{d}x - \lambda \int_\Omega uv\mathrm{d}x - \int_\Omega |u|^{\alpha-2}uv\mathrm{d}x, \\ \langle F(u), v\rangle &:= \int_\Omega f(x,u)v\mathrm{d}x, \end{aligned} \tag{2}$$

where $\langle .,.\rangle$ stands for the duality pairing between $\mathbf{X}^*$ and $\mathbf{X}$.

It was well known that the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a sequence of eigenvalues $\{\lambda_k\}$ with $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. Moreover, there exists a sequence of eigenfunctions $\{\varphi_k\}$ corresponding to the eigenvalues $\{\lambda_k\}$ of (3) such that

$$\begin{aligned} &\|\varphi_k\|^2 = 1 = \lambda_k \|\varphi_k\|^2_{L^2(\Omega)}, \quad k = 1, 2, \ldots, \\ &\int_\Omega \nabla\varphi_k(x)\nabla\varphi_j(x)\mathrm{d}x = \int_\Omega \varphi_k(x)\varphi_j(x)\mathrm{d}x = 0 \quad \text{provided } k \neq j \end{aligned} \tag{4}$$

and

$$H^1_0(\Omega) = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_k) \oplus \cdots, \tag{5}$$

where $E(\lambda_k)$ denotes the finite dimensional space of eigenfunctions corresponding to λ_k. It is clear that for a given $k \in \mathbb{N}$, there exists an $s_0 > 0$ such that

$$\int_\Omega u^2 \,\mathrm{d}x \geq s_0 \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \quad \text{for all } u \in E(\lambda_1) \oplus \cdots \oplus E(\lambda_k). \tag{6}$$

The main results that we obtain in this paper are:

Theorem 1.1. *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be continuous, satisfying the following (f_1)–(f_3):*

(f_1) $f(x,t) < 0$ *for all* $(x,t) \in \overline{\Omega} \times (0,+\infty)$ *and* $f(x,t) > 0$ *for all* $(x,t) \in \overline{\Omega} \times (-\infty, 0)$*;*

(f_2) $\lim_{t\to 0} \frac{f(x,t)}{t^p} = -l_1$ *uniformly in* $\overline{\Omega}$, *where* $0 < p := r_1/s_1 < 1$ *with* r_1 *and* s_1 *being odd,* $l_1 > 0$*;*
(f_3) $\lim_{t\to\infty} \frac{f(x,t)}{t^q} = -l_2$ *uniformly in* $\overline{\Omega}$, *where* $1 < q := r_2/s_2 < 2^* - 1$ *with* r_2 *and* s_2 *being odd,* $l_2 > 0$, $2^* = 2N/(N-2)$ *is the critical exponent for the Sobolev embedding* $\mathbf{X} \hookrightarrow L^p(\Omega)$.

Suppose that β *is a penalty operator associated with* $\mathbf{K}$. *Then there exists a sequence* $\{u_\varepsilon\} \subset \mathbf{X}$ *which converges in* $\mathbf{K}$ *toward a solution of Problem (1) with* $\lambda = \lambda_1$ *as* $\varepsilon \to 0^+$, *where for each* $\varepsilon > 0$, $u_\varepsilon \in \mathbf{X}$ *satisfies the following penalized equation*

$$A(u_\varepsilon) + \frac{1}{\varepsilon}\beta(u_\varepsilon) = F(u_\varepsilon). \tag{7}$$

Remark 1.1. (i) Obviously, functions satisfying $(f_1)-(f_3)$ exist. For instance, let $\Omega \subset \mathbb{R}^3$, the functions $f(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,t) = \begin{cases} -\left(|t| \prod_{i=1}^{3} \sin x_i + 2\right) t^{1/3} & \text{if } x = (x_1,x_2,x_3) \in \overline{\Omega} \text{ and } |t| \le 1 \\ -\left(|t|^{-1} \prod_{i=1}^{3} \sin x_i + 2\right) t^{5/3} & \text{if } x = (x_1,x_2,x_3) \in \overline{\Omega} \text{ and } |t| > 1 \end{cases}$$

satisfies $(f_1)-(f_3)$.
(ii) The existence of solutions for problems like (1) with $0 < \lambda < \lambda_1$ was known, see for example [8].

Theorem 1.2. *Suppose that* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is continuous and* f *satisfies* (f_1) *and*

$$(f_2') \quad -l_3 < \lim_{t\to\infty} \frac{f(x,t)}{t} < -l_4,$$

where $l_3 > l_4 > \frac{1}{s_0}(\frac{\lambda_{k+1}}{\lambda_1} - 1)$, s_0 *is as in (6). Let* β *be a penalty operator associated with* $\mathbf{K}$. *Suppose that* $\lambda_k < \lambda_{k+1}$ *for some* $k \in \mathbb{N}$ *and* $\lambda_k < \lambda < \lambda_{k+1}$. *Then there exists a sequence* $\{u_\varepsilon\}$, *where each* $u_\varepsilon \in \mathbf{X}$, *satisfying the penalized equation (7), which converges weakly in* $\mathbf{K}$ *toward a solution of Problem (1) as* $\varepsilon \to 0^+$.

Remark 1.2. Under different assumptions, by using the linking technique, an existence result for an elliptic variational–hemivariational inequality where $\lambda_k < \lambda < \lambda_{k+1}$ was obtained by Motreanu and Rădulescu (cf. [2, Theorem 3.3]).

2. Proofs the main results

Proof of Theorem 1.1. Denote $E(\lambda_1)^\perp := E(\lambda_2) \oplus \cdots \oplus E(\lambda_k) \oplus \cdots$; then by (5), for each $u \in H_0^1(\Omega)$, there exist $u_1 \in E(\lambda_1)$, $u_2 \in E(\lambda_1)^\perp$ such that $u = u_1 + u_2$, and hence

$$\begin{aligned} &\int_\Omega |\nabla u_1|^2 \,\mathrm{d}x = \lambda_1 \int_\Omega u_1^2 \,\mathrm{d}x, \\ &\int_\Omega |\nabla u_2|^2 \,\mathrm{d}x \ge \lambda_2 \int_\Omega u_2^2 \,\mathrm{d}x. \end{aligned} \tag{8}$$

Since

$$\langle A(u), u\rangle = \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda_1 \int_\Omega u^2 \,\mathrm{d}x - \int_\Omega |u|^\alpha \,\mathrm{d}x,$$

it follows from (8) and the Sobolev inequality, that there exists a positive constant c_1 such that

$$\begin{aligned} \langle A(u), u\rangle &\ge \int_\Omega |\nabla u_1|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_2|^2 \,\mathrm{d}x - \lambda_1 \int_\Omega u_1^2 \,\mathrm{d}x - \lambda_1 \int_\Omega u_2^2 \,\mathrm{d}x - c_1 \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^{\alpha/2} \\ &\ge \int_\Omega |\nabla u_2|^2 \,\mathrm{d}x - \frac{\lambda_1}{\lambda_2} \int_\Omega |\nabla u_2|^2 \,\mathrm{d}x - c_1 \|u\|^\alpha \\ &= \left(1 - \frac{\lambda_1}{\lambda_2}\right) \|u_2\|^2 - c_1 \|u\|^\alpha. \end{aligned} \tag{9}$$

By (f_2), $\forall \varepsilon : 0 < \varepsilon < l_1$ there exists $0 < \delta_\varepsilon < 1$ such that $\forall t :\ 0 < |t| < \delta_\varepsilon, \forall x \in \overline{\Omega}$,

$$\left| \frac{f(x,t)}{t^p} + l_1 \right| < \varepsilon.$$

It follows that if $(x,t) \in \overline{\Omega} \times [0, \delta_\varepsilon]$ then

$$f(x,t) \le (\varepsilon - l_1)t^p \le (\varepsilon - l_1)t; \tag{10}$$

and if $(x,t) \in \overline{\Omega} \times [-\delta_\varepsilon, 0]$, then

$$f(x,t) \ge (\varepsilon - l_1)t^p \ge (\varepsilon - l_1)t. \tag{11}$$

Similarly, by (f_3), $\forall\ \varepsilon : 0 < \varepsilon - l_2 < 0$ there exists an $M_\varepsilon > 1$ such that if $(x,t) \in \overline{\Omega} \times [M_\varepsilon, +\infty)$, then

$$f(x,t) \le (\varepsilon - l_2)t^q \le (\varepsilon - l_2)t; \tag{12}$$

if $(x,t) \in \overline{\Omega} \times (-\infty, M_\varepsilon]$, then

$$f(x,t) \ge (\varepsilon - l_2)t^q \ge (\varepsilon - l_2)t. \tag{13}$$

Since $f(x,t)$ is continuous on $\overline{\Omega} \times [\delta_\varepsilon, M_\varepsilon]$ and $f(x,t)$ satisfies (f_1), there exists $(x_\varepsilon, t_\varepsilon) \in \overline{\Omega} \times [\delta_\varepsilon, M_\varepsilon]$ such that $\forall\ (x,t) \in \overline{\Omega} \times [\delta_\varepsilon, M_\varepsilon]$,

$$f(x,t) \le f(x_\varepsilon, t_\varepsilon) < 0. \tag{14}$$

Then, $\forall\ (x,t) \in \overline{\Omega} \times [\delta_\varepsilon, M_\varepsilon]$,

$$f(x,t) \le \frac{f(x_\varepsilon, t_\varepsilon)}{M_\varepsilon} t. \tag{15}$$

Similarly, there exists $(x'_\varepsilon, t'_\varepsilon) \in \overline{\Omega} \times [-M_\varepsilon, -\delta_\varepsilon]$ such that $\forall\ (x,t) \in \overline{\Omega} \times [-M_\varepsilon, -\delta_\varepsilon]$,

$$f(x,t) \ge -\frac{f(x'_\varepsilon, t'_\varepsilon)}{M_\varepsilon} t. \tag{16}$$

It follows from (10)–(16) that there exist positive constants c_2 and c_3 such that

$$f(x,t) \le -c_2 t, \quad \forall\ (x,t) \in \overline{\Omega} \times [0, +\infty),$$

and

$$f(x,t) \ge -c_3 t, \quad \forall\ (x,t) \in \overline{\Omega} \times (-\infty, 0].$$

Therefore, there exists a constant $c_4 > 0$ such that

$$\langle F(u), u \rangle \le -c_4 \int_\Omega u^2\, \mathrm{d}x. \tag{17}$$

The monotonicity of β and $\beta(0) = 0$ imply that

$$\langle \beta(u), u \rangle \ge 0. \tag{18}$$

It follows from (9), (17) and (18) and the first equation of (8) that

$$\begin{aligned}
\langle A(u), u \rangle + \frac{1}{\varepsilon}\langle \beta(u), u \rangle - \langle F(u), u \rangle &\ge \left(1 - \frac{\lambda_1}{\lambda_2}\right) \|u_2\|^2 - c_1 \|u\|^\alpha + c_4 \int_\Omega u^2\, \mathrm{d}x \\
&= \left(1 - \frac{\lambda_1}{\lambda_2}\right) \int_\Omega |\nabla u_2|^2\, \mathrm{d}x - c_1 \|u\|^\alpha + c_4 \lambda_1 \int_\Omega |\nabla u_1|^2\, \mathrm{d}x \\
&\ge c_5 \int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2)\, \mathrm{d}x - c_1 \|u\|^\alpha \\
&= c_5 \|u\|^2 - c_1 \|u\|^\alpha,
\end{aligned}$$

where $c_5 = \min\{(1 - \lambda_1/\lambda_2), c_4\lambda_1\}$. Therefore,

$$\frac{\langle A(u), u\rangle + \frac{1}{\varepsilon}\langle \beta(u), u\rangle - \langle F(u), u\rangle}{\|u\|} \to \infty \quad \text{as } \|u\| \to \infty. \tag{19}$$

We claim that for each $\varepsilon > 0$, $A + \frac{1}{\varepsilon}\beta - F$ is a mapping of $(S)_+$. In fact, suppose that $\{u_k\} \subset \mathbf{X}$ satisfies $u_k \rightharpoonup u_0 \in \mathbf{X}$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \left\langle A(u_k) + \frac{1}{\varepsilon}\beta(u_k) - F(u_k), u_k - u_0 \right\rangle \le 0.$$

Since the embedding $\mathbf{X} \hookrightarrow L^2(\Omega)$ is compact, it follows that $u_k \to u_0 \in L^2(\Omega)$, and then $u_k(x) \to u_0(x)$ a.e. in Ω as $k \to \infty$. By (f_2) and (f_3), there exist positive constants c_6, c_7 and c_8 such that $|f(x,t)| \le c_6|t|^p + c_7|t|^q + c_8$. Hence, $\int_\Omega f(x, u_k)u_k \to \int_\Omega f(x, u_0)u_0$ as $k \to \infty$. Thus, we have

$$\begin{aligned}
\limsup_{k\to\infty} \int_\Omega \nabla u_k \nabla(u_k - u_0)\mathrm{d}x &\le \limsup_{k\to\infty} \langle A(u_k), u_k - u_0\rangle \\
&\quad + \limsup_{k\to\infty} \lambda \int_\Omega u_k(u_k - u_0)\mathrm{d}x + \limsup_{k\to\infty} \int_\Omega |u_k|^{\alpha-2}(u_k - u_0)\mathrm{d}x \\
&\le \limsup_{k\to\infty} \left\langle A(u_k) + \frac{1}{\varepsilon}\beta(u_k) - F(u_k), u_k - u_0 \right\rangle \\
&\quad + \limsup_{k\to\infty} \left\langle -\frac{1}{\varepsilon}\beta(u_k), u_k - u_0 \right\rangle + \limsup_{k\to\infty} \langle F(u_k), u_k - u_0\rangle \\
&\le -\frac{1}{\varepsilon} \liminf_{k\to\infty} \int_\Omega \beta(u_0)(u_k - u_0)\mathrm{d}x + \limsup_{k\to\infty} \int_\Omega f(x, u_k)(u_k - u_0)\mathrm{d}x = 0,
\end{aligned}$$

this gives that $\|u_k\| \to \|u\|$ as $k \to \infty$. Therefore, $u_k \to u_0 \in \mathbf{X}$ as $k \to \infty$ since $\mathbf{X}$ is uniformly convex. Thus, we have proved that for each $\varepsilon > 0$, $A + \frac{1}{\varepsilon}\beta - F$ is a mapping of $(S)_+$. By Theorem 2.2 in [9], for each small ε, the equation

$$A(u_\varepsilon) + \frac{1}{\varepsilon}\beta(u_\varepsilon) = F(u_\varepsilon) \tag{20}$$

admits at least one solution $u_\varepsilon \in \mathbf{X}$. The coercive property (19) implies the existence of a constant C, independent of the choice of $\varepsilon > 0$, such that $\|u_\varepsilon\| \le C$. Hence, we may choose a sequence $\{\varepsilon(n)\}$, such that $\varepsilon(n) \to 0^+$ and $\widehat{u}_n = u_{\varepsilon(n)} \rightharpoonup u' \in \mathbf{X}$ as $n \to \infty$. It is easy to prove that $\{A(\widehat{u}_n) - F(\widehat{u}_n)\}$ is also bounded in $\mathbf{X}$, and then

$$\beta(\widehat{u}_n) = \varepsilon(n)[F(\widehat{u}_n) - A(\widehat{u}_n)] \to 0$$

as $n \to \infty$. Now for each $v \in \mathbf{X}$, we have

$$\begin{aligned}
\langle \beta(v), u' - v\rangle &= \lim_{n\to\infty} \langle \beta(v), \widehat{u}_n - v\rangle \\
&\le \limsup_{n\to\infty} \langle \beta(v) - \beta(\widehat{u}_n), \widehat{u}_n - v\rangle + \limsup_{n\to\infty} \langle \beta(\widehat{u}_n), \widehat{u}_n - v\rangle \\
&\le 0,
\end{aligned}$$

which yields $\langle \beta(u' - tw), w\rangle \le 0$ with the choice of $v = u' - tw$ with $t > 0$ and $w \in \mathbf{X}$. By the hemicontinuity of β, we have $\langle \beta(u'), w\rangle \le 0$, and hence $\beta(u') = 0$, i.e., $u' \in \mathbf{K}$. Since $A + \frac{1}{\varepsilon}\beta - F$ is a mapping of $(S)_+$, (20) implies that $\widehat{u}_n \to u'$ as $n \to \infty$.

Let $\omega \in \mathbf{K}$; then $\beta(\omega) = 0$, and

$$\langle A(\widehat{u}_n) - F(\widehat{u}_n), \omega - \widehat{u}_n\rangle = +\frac{1}{\varepsilon(n)} \langle \beta(\omega) - \beta(\widehat{u}_n), \omega - \widehat{u}_n\rangle \ge 0.$$

Hence

$$\langle A(u') - F(u'), \omega - u'\rangle \ge 0.$$

That is, u' is a solution of (1). ■

Proof of Theorem 1.2. For each given $k \in \mathbb{N}$, set $\mathbf{V} = E(\lambda_1) \oplus \cdots \oplus E(\lambda_k)$ and $\mathbf{W} = \mathbf{V}^{\perp}$; then it follows from (5) that for each $u \in \mathbf{X}$ there exist $u_V \in \mathbf{V}$ and $u_W \in \mathbf{W}$ such that $u = u_V + u_W$. Since $u_V \in \mathbf{V}$ can be written as $u_V = \sum_{i=1}^{k} t_i \varphi_i$ ($t_1, \ldots, t_k \in \mathbb{R}$), we obtain, by (4) that

$$\int_\Omega |\nabla u_V|^2 \,\mathrm{d}x = \sum_{i=1}^{k} t_i \lambda_i \|\varphi_i\|^2_{L^2(\Omega)}.$$

Hence

$$\begin{aligned} &\int_\Omega |\nabla u_V|^2 \,\mathrm{d}x \geq \lambda_1 \int_\Omega u_V^2 \,\mathrm{d}x, \\ &\int_\Omega |\nabla u_W|^2 \,\mathrm{d}x \geq \lambda_{k+1} \int_\Omega u_W^2 \,\mathrm{d}x. \end{aligned} \tag{21}$$

By (21) and the Sobolev inequality, there exists a positive constant s_1 such that

$$\begin{aligned} \langle A(u), u\rangle &\geq \int_\Omega |\nabla u_V|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_W|^2 \,\mathrm{d}x - \lambda \int_\Omega u_V^2 \,\mathrm{d}x - \lambda \int_\Omega u_W^2 \,\mathrm{d}x - s_1 \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x \right)^{\alpha/2} \\ &\geq \left(1 - \frac{\lambda}{\lambda_1}\right) \int_\Omega |\nabla u_V|^2 \,\mathrm{d}x + \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \int_\Omega |\nabla u_W|^2 \,\mathrm{d}x - s_1 \|u\|^\alpha \\ &= \left(1 - \frac{\lambda}{\lambda_1}\right) \|u_V\|^2 + \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|u_W\|^2 - s_1 \|u\|^\alpha. \end{aligned} \tag{22}$$

It follows from (f_2') that there exists an $M > 0$ such that

$$\frac{f(x,t)}{t} \leq -l_4, \quad \forall |t| > M.$$

The continuity of $f(x, u)$ implies that there exists a constant $s_2 > 0$ such that

$$\langle F(u), u\rangle \leq -l_4 \int_\Omega u^2 \,\mathrm{d}x + s_2. \tag{23}$$

By (22), (6) and (23), we have

$$\begin{aligned} \langle A(u), u\rangle + \frac{1}{\varepsilon}\langle \beta(u), u\rangle - \langle F(u), u\rangle &\geq \left(1 - \frac{\lambda}{\lambda_1}\right) \|u_V\|^2 + \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|u_W\|^2 - s_1 \|u\|^\alpha \\ &\quad + \left(\frac{\lambda_{k+1}}{\lambda_1} - 1\right) \int_\Omega |\nabla u_V|^2 \,\mathrm{d}x - s_2 \\ &\geq \left(\frac{\lambda_{k+1} - \lambda}{\lambda_1}\right) \|u_V\|^2 + \left(\frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}}\right) \|u_W\|^2 - s_1 \|u\|^\alpha - s_2 \\ &\geq \left(\frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}}\right) \|u\|^2 - s_1 \|u\|^\alpha - s_2, \end{aligned}$$

which gives the coerciveness of $A + \frac{1}{\varepsilon}\beta - F$ in the sense of (19). By using the same arguments after (19) in the proof of Theorem 1.1, we obtain the required conclusion. ■

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